Asymmetric Stress Analysis of Axisymmetric Solids with Rectangularly Orthotropic Properties

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An analysis technique is presented to perform the thermal stress analysis of axisymmetric solids constructed of a rectangularly orthotropic material and loaded axi- or asymmetrically. Specifically the modifications are presented for a solid of revolution computer code, ASAAS, which approximates the asymmetric stress-strain behavior by expanding the problem asymmetries in Fourier harmonic form. The modifications required to analyze a rectangularly orthotropic material are associated primarily with the stiffness matrix and thermal load vector calculations. The comparison of two numerical examples with the theoretical results indicates an excellent correlation for both stress and strain quantities throughout the body. The increase in computer code capability is particularly significant when considering the thermal stress analyses of re-entry vehicles made of three-dimensional carbon-carbon materials.

I. Introduction

THE ubiquitous solid-of-revolution stress analyses computer codes[1-4] usually are limited to modeling material behavior as either transversely isotropic or polar orthotropic. Variations of these basic material models accommodate materials with different tension and compression properties, as well as material property axes noncoincident with the R-Z axes. Despite this sophistication in material modeling, a technical void still exists. Specifically, the recent interest in three-dimensional carbon-carbon composite materials (such as Fig. 1) for re-entry vehicle nosetip applications has generated a need for a rectangularly orthotropic elastic material model. Such a model is neither transversely isotropic nor polar orthotropic.

To model the macroscopic behavior of a rectangularly orthotropic composite material a three-dimensional finite-element code could be used at considerable cost. The purpose of this work is to demonstrate the modifications to a well-documented two-dimensional code to enable the low cost analysis of three-dimensional rectangularly orthotropic materials fabricated into axisymmetric shapes (see Fig. 2) and loaded axi- or asymmetrically. Specifically, the modifications are presented for the ASAAS[2,4] structural analyses code. A similar (but approximate) technique was described previously[5] for two-dimensional axisymmetric codes.

The ASAAS code originally was formulated to accommodate asymmetric material property variations and asymmetric loading of axisymmetric bodies. It was implied that the circumferential variations of properties were due to a temperature-dependent material being subjected to an asymmetric thermal environment. The inclusion of a rectangularly orthotropic material also implies a circumferential variation of properties, but this variation results from the rectangular to polar coordinate transformation of elastic properties. In this case, substantial ASAAS code modifications required to analyze a rectangularly orthotropic material are associated primarily with the stiffness matrix and thermal load vector.

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The modification to the stiffness matrix rests on the fact that when the rectangular orthotropic elastic properties are transformed to polar coordinates they can be expressed exactly as a truncated Fourier series. Whereas the Fourier

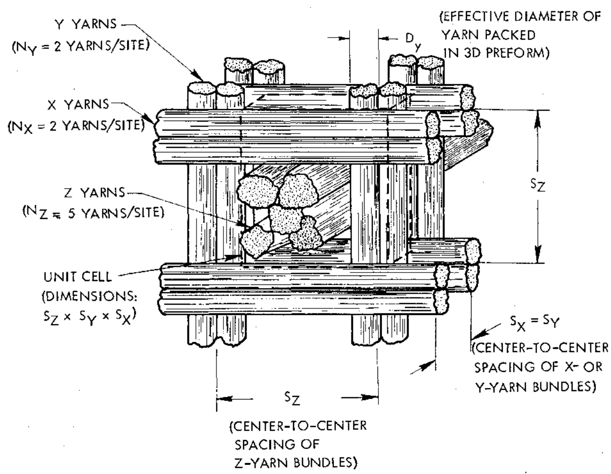

Fig. 1 Three-dimensional preform geometrical parameters.

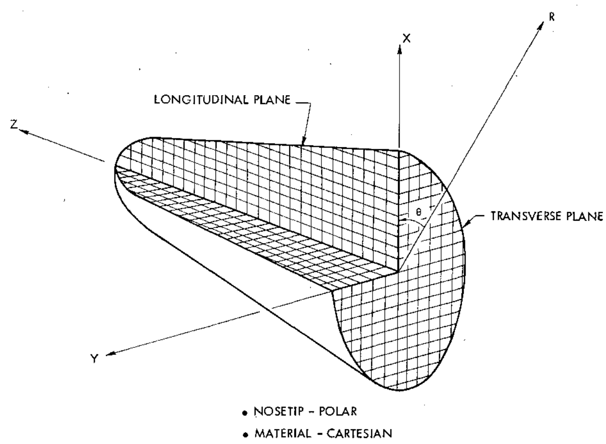

Fig. 2 Schematic relationship between nosetip coordinate system and material coordinate system.

series coefficients within the elasticity matrix for transversely isotropic or polar orthotropic materials are independent of θ, there is additional harmonic coupling for a rectangularly orthotropic material due to the polar coordinate transformation terms of $\sin 2\theta, \sin 4\theta, \ldots, \cos 4\theta$. The specific modifications to the ASAAS code for the stiffness matrix entail a redefinition[2] of the 9×9 matrix $\phi_{m_\ell m_d}$ and a new algorithm for expressing the harmonic coupling of load, displacement, material, and polar coordinate transformation terms.

The changes to the thermal load vector for a rectangularly orthotropic material are considerably simpler than the modification to the stiffness matrix. The thermal stress vector, which is used to compute the thermal load, is derived from the product of the elasticity matrix and the thermal strain vector. Since the resulting thermal stress vector has an additional harmonic term due to material orthotropy, special attention is given to the proper coupling of load, material, and polar coordinate transformation terms.

II. ASAAS Code Modifications

A. Elasticity Matrix

The ASAAS code solves the thermostructural response problem for solids-of-revolution subjected to asymmetric surface traction and temperature distributions. Since the variation of material properties with temperature is included within the formulation, asymmetric material property capability is inherent in the program. The analysis is formulated in polar coordinates; thus, all variations of loads, displacements, and material properties in the circumferential direction are represented conveniently by a truncated Fourier series. If, however, there is no circumferential variation of temperatures, the material properties as well as the element stiffness matrices are independent of the circumferential coordinate θ.

In the original version of ASAAS, material models of isotropic, transversely isotropic, and polar orthotropic properties were accommodated readily by expressing the elastic coefficients in matrix form as

$$E=\sum_{m_m=0}^{M_m} E_{m_m}\cos m_m\theta \qquad (1)$$

where

E = elasticity matrix (6×6)
E_{m_m} = matrix of Fourier coefficients for the elastic constants (6×6)
m_m = material harmonic number
M_m = material harmonic truncation level

The usual technique in constructing an element stiffness matrix is to express the strain energy of the ith element in matrix quadratic form:

$$U_i=\tfrac{1}{2}\int_{v_i}\epsilon_i^T\sigma_i\mathrm{d}v_i=\tfrac{1}{2}\int_{v_i}\epsilon_i^TE_i\epsilon_i\mathrm{d}v_i=\tfrac{1}{2}D_i^TK_iD_i \qquad (2)$$

where

U_i = strain energy of ith element
ϵ_i = vector of element strains ($m\times1$)
σ_i = vector of element stresses ($m\times1$)
D_i = vector of element nodal displacements ($n\times1$)
E_i = element elasticity matrix ($m\times m$)
K_i = element stiffness matrix ($n\times n$)

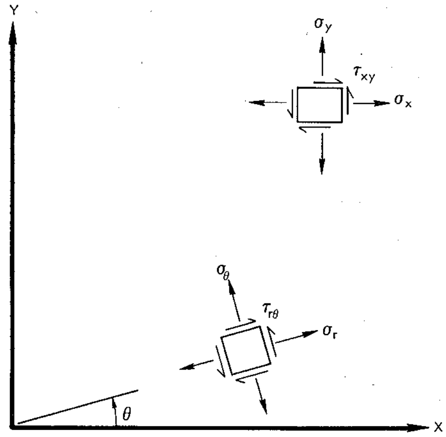

Fig. 3 Definition of terms for rectangular to polar transformation.

For a solid-of-revolution the volume integral is expressed in polar coordinate form as

$$U_i=\tfrac{1}{2}\int_{v_i}\epsilon_i^TE_i\epsilon_i\mathrm{d}v_i=\tfrac{1}{2}\int_{A_i}\int_0^{2\pi}\epsilon_i^TE_i\epsilon_ir\mathrm{d}\theta\,\mathrm{d}A_i \qquad (3)$$

Since the primary objective of the present discussion is to incorporate the representation of a material into equations that are inherently polar, it is necessary to consider the transformation of the element elasticity matrix E_i from rectangular to polar coordinates. For instance, the transformation of stresses from rectangular XYZ to cylindrical $R\theta Z$ coordinates,[6] shown schematically in Fig. 3, can be expressed by

$$\begin{Bmatrix}\sigma_R\\ \sigma_Z\\ \sigma_\theta\\ \tau_{RZ}\\ \tau_{R\theta}\\ \tau_{Z\theta}\end{Bmatrix}=\begin{bmatrix}\cos^2\theta & 0 & \sin^2\theta & 0 & 2\cos\theta\sin\theta & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \sin^2\theta & 0 & \cos^2\theta & 0 & -2\cos\theta\sin\theta & 0\\ 0 & 0 & 0 & \cos\theta & 0 & \sin\theta\\ -\cos\theta\sin\theta & 0 & \cos\theta\sin\theta & 0 & \cos^2\theta-\sin^2\theta & 0\\ 0 & 0 & 0 & -\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\sigma_X\\ \sigma_Z\\ \sigma_Y\\ \tau_{XZ}\\ \tau_{XY}\\ \tau_{ZY}\end{bmatrix} \qquad (4)$$

or more compactly as

$$\sigma_{R\theta Z}=T\sigma_{XYZ} \qquad (5)$$

Similarly,[6]

$$\epsilon_{XYZ}=T^T\epsilon_{R\theta Z} \qquad (6)$$

The stress and strain transformation equations then may be used to transform the constitutive relations from rectangular XYZ coordinates to cylindrical $R\theta Z$ as follows:

$$\sigma_{XYZ}=E_{XYZ}\epsilon_{XYZ} \qquad (7a)$$

$$\sigma_{R\theta Z}=TE_{XYZ}\epsilon_{XYZ}=TE_{XYZ}T^T\epsilon_{R\theta Z} \qquad (7b)$$

$$\sigma_{R\theta Z}=E_{R\theta Z}\epsilon_{R\theta Z} \qquad (7c)$$

where the elasticity matrix of a rectangularly orthotropic material expressed in polar coordinates satisfies the relation

$$E_{R\theta Z}=TE_{XYZ}T^T \qquad (8)$$

The elastic stiffness matrix of a rectangularly orthotropic material may be expressed symbolically as

$$E_{XYZ}=\begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{66} \end{bmatrix} \tag{9}$$

where the zero terms are noted explicitly.

When the rectangularly orthotropic material properties are transformed from Cartesian coordinates to polar coordinates, the material property matrix takes on several additional nonzero terms as indicated by the following expression:

$$E_{R\theta Z}=\begin{bmatrix} E^*_{11} & E^*_{12} & E^*_{13} & 0 & E^*_{15} & 0 \\ E^*_{12} & E^*_{22} & E^*_{23} & 0 & E^*_{25} & 0 \\ E^*_{13} & E^*_{23} & E^*_{33} & 0 & E^*_{35} & 0 \\ 0 & 0 & 0 & E^*_{44} & 0 & E^*_{46} \\ E^*_{15} & E^*_{25} & E^*_{35} & 0 & E^*_{55} & 0 \\ 0 & 0 & 0 & E^*_{46} & 0 & E^*_{66} \end{bmatrix} \tag{10}$$

Term-by-term expansion of Eq. (10) produces the following detailed relationships between the Cartesian and polar terms. Note that trigonometric identities have been used to express the results in a more convenient form.

$$E^*_{11}=\tfrac{1}{8}[4E_{55}+2E_{13}+3(E_{11}+E_{33})]+\tfrac{1}{2}[E_{11}-E_{33}]\cos 2\theta$$
$$+\tfrac{1}{8}[E_{11}+E_{33}-4E_{55}-2E_{13}]\cos 4\theta \tag{11a}$$

$$E^*_{12}=\tfrac{1}{2}[E_{12}+E_{23}]+\tfrac{1}{2}[E_{12}-E_{23}]\cos 2\theta \tag{11b}$$

$$E^*_{13}=\tfrac{1}{8}[E_{11}+E_{33}+6E_{13}-4E_{55}]$$
$$-\tfrac{1}{8}[E_{11}+E_{33}-2E_{13}-4E_{55}]\cos 4\theta \tag{11c}$$

$$E^*_{15}=\tfrac{1}{4}[E_{33}-E_{11}]\sin 2\theta$$
$$-\tfrac{1}{8}[E_{11}+E_{33}-4E_{55}-2E_{13}]\sin 4\theta \tag{11d}$$

$$E^*_{22}=E_{22} \tag{11e}$$

$$E^*_{23}=\tfrac{1}{2}[E_{12}+E_{23}]+\tfrac{1}{2}[E_{23}-E_{12}]\cos 2\theta \tag{11f}$$

$$E^*_{25}=\tfrac{1}{2}[E_{23}-E_{12}]\sin 2\theta \tag{11g}$$

$$E^*_{33}=\tfrac{1}{8}[4E_{55}+2E_{13}+3(E_{11}+E_{33})]+\tfrac{1}{2}[E_{33}-E_{11}]\cos 2\theta$$
$$+\tfrac{1}{8}[E_{11}+E_{33}-4E_{55}-2E_{13}]\cos 4\theta \tag{11h}$$

$$E^*_{35}=\tfrac{1}{4}[E_{33}-E_{11}]\sin 2\theta$$
$$+\tfrac{1}{8}[E_{11}+E_{33}-4E_{55}-2E_{13}]\sin 4\theta \tag{11i}$$

$$E^*_{44}=\tfrac{1}{2}[E_{44}+E_{66}]+\tfrac{1}{2}[E_{44}-E_{66}]\cos 2\theta \tag{11j}$$

$$E^*_{46}=\tfrac{1}{2}[E_{66}-E_{44}]\sin 2\theta \tag{11k}$$

$$E^*_{55}=\tfrac{1}{8}[E_{11}+E_{33}-2E_{13}+4E_{55}]$$
$$-\tfrac{1}{8}[E_{11}+E_{33}-2E_{13}-4E_{55}]\cos 4\theta \tag{11l}$$

$$E^*_{66}=\tfrac{1}{2}[E_{44}+E_{66}]+\tfrac{1}{2}[E_{66}-E_{44}]\cos 2\theta \tag{11m}$$

Since it is desired to model the circumferential variation of material stiffness (e.g., due to temperature variation and material property temperature dependence), the polar coordinate elasticity matrix $\boldsymbol{E}_{R\theta Z}$ [Eqs. (10) and (11)] can be substituted for $\boldsymbol{E}_{m_m}$ in Eq. (1). Thus

$$\boldsymbol{E}_{m_m}=\boldsymbol{E}_{R\theta Z} \tag{12}$$

$$\boldsymbol{E}=\sum_{m_m=0}^{M_m}\sum_{n=0,2}^{4}(\boldsymbol{A}_n\sin n\theta+\boldsymbol{B}_n\cos n\theta)\cos m_m\theta \tag{13}$$

This expression presents the complete description of circumferentially varying material properties of a rectangularly orthotropic material in polar coordinates.

B. Stiffness Matrix

Inasmuch as the components of the element strain vector ϵ also are assumed to vary circumferentially, the volume integral for the stiffness matrix may be represented by

$$\boldsymbol{K}=\sum_{m_\ell=0}^{M_\ell}\sum_{m_d=0}^{M_d}\boldsymbol{h}^T\boldsymbol{\phi}_{m_\ell m_d}\boldsymbol{h} \tag{14}$$

where

$$\boldsymbol{\phi}_{m_\ell m_d}\equiv\left\{\int_{A_i}\int_0^{2\pi}\boldsymbol{g}^T_{m_\ell}\boldsymbol{E}_{m_m}\boldsymbol{g}_{m_d}r\,d\theta\, dA\right\}$$

The matrices $\boldsymbol{h}$ and $\boldsymbol{g}$ relate the element strains to nodal displacements with matrix $\boldsymbol{h}$ constant for the element volume and matrix $\boldsymbol{g}$ variable over the element volume. The subscripts m_ℓ, m_d, and m_m refer to load, displacement, and material harmonics, respectively. The 9×9 matrix $\boldsymbol{h}$ is an assemblage of geometrical constants, and is therefore the same for a solid-of-revolution constructed of an isotropic, polar orthotropic, or rectangularly orthotropic material. The 9×9 matrix $\boldsymbol{\phi}$, on the other hand, is altered depending upon the material model being used. Thus, changes to the element stiffness matrix to account for a rectangularly orthotropic material are the direct result of modifications to the matrix $\boldsymbol{\phi}$. These modifications are the result of incorporating the 6×6 matrix of material property Fourier coefficients, $\boldsymbol{E}_m$, in the derivation of $\boldsymbol{\phi}$. The $\boldsymbol{A}_n$, $\boldsymbol{B}_n$, $\boldsymbol{\phi}$, $\boldsymbol{h}$, and $\boldsymbol{g}$ matrices are described in detail in the Appendix.

In summary, the modification to the stiffness matrix rests on the fact that when rectangular orthotropic elastic relations are transformed to polar coordinates they can be expressed exactly as a truncated Fourier series. Whereas the Fourier coefficients within the $\boldsymbol{E}_m$ matrix for transversely isotropic or polar orthotropic materials are independent of θ, it can be seen from Eq. (13) that there is additional harmonic coupling for a rectangularly orthotropic material due to the polar coordinate transformation terms ($\sin 2\theta$, $\sin 4\theta$, $\cos 4\theta$).

C. Thermal Load Vector

The changes to the thermal load vector for a rectangularly orthotropic material are considerably simpler than the modifications to the stiffness matrix. Briefly, the thermal stress vector, which is used to compute loads, is derived from the product of the elasticity matrix and the thermal strain vector. The thermal stress in polar coordinates due to complete restraint of thermal expansion is

$$\boldsymbol{\sigma}_{R\theta Z}=\boldsymbol{E}_{R\theta Z}\boldsymbol{\alpha}_{R\theta Z}\Delta t \tag{15}$$

where

$$\boldsymbol{\alpha}_{XYZ}=\boldsymbol{T}^T\boldsymbol{\alpha}_{R\theta Z} \tag{16}$$

and $\boldsymbol{E}_{R\theta Z}$ is defined by Eq. (8).

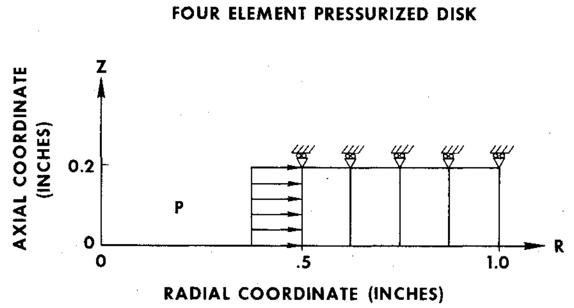

Fig. 4 Finite element idealization.

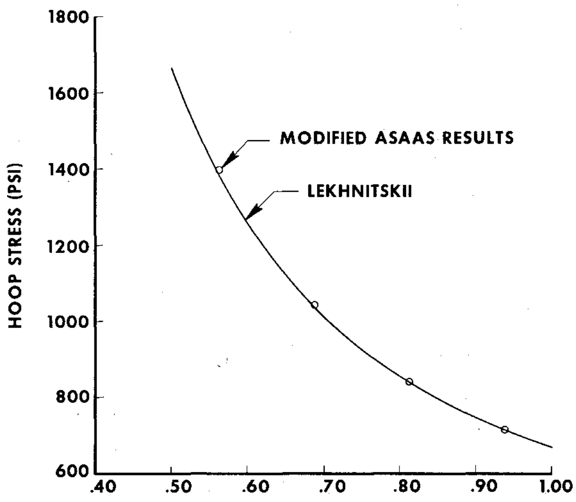

Fig. 5 Hoop stress in pressurized orthotropic disk.

Equation (16) easily may be inverted by substituting $-\theta$ for $+\theta$ in T. The result is

$$\alpha_{R\theta Z}=\begin{Bmatrix} \frac{1}{2}(\alpha_X+\alpha_Y)+\frac{1}{2}(\alpha_X-\alpha_Y)\cos 2\theta \\ \alpha_Z \\ \frac{1}{2}(\alpha_X+\alpha_Y)+\frac{1}{2}(\alpha_Y-\alpha_X)\cos 2\theta \\ 0 \\ (\alpha_Y-\alpha_X)\sin 2\theta \\ 0 \end{Bmatrix} \tag{17}$$

III. Example Problems

A. Example 1: Pressurized Disk

In order to verify the accuracy of the ASAAS code modifications, the four-element stress analysis of a pressurized rectangularly orthotropic disk (Fig. 4) was investigated. Although there does not appear to be a closed-form solution within the literature for general rectangular orthotropy, Ref. 7 does indicate the solution for a specific choice of material properties. An axisymmetric stress state will be assured[7] if the material properties are chosen such that

$$((E_x/G_{xy})-2\nu_{xy})-(E_x/E_y)=1 \tag{18}$$

Furthermore, the theoretical stress state is identical to that of an isotropic pressurized disk (plane stress solution)

$$\sigma_r=\frac{p}{(a/b)^2-1}\{1-(a/r)^2\} \tag{19a}$$

$$\sigma_\theta=\frac{p}{(a/b)^2-1}\{1+(a/r)^2\} \tag{19b}$$

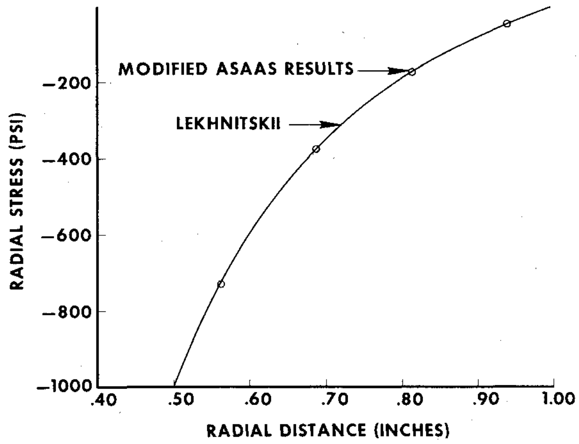

Fig. 6 Radial stress in pressurized orthotropic disk.

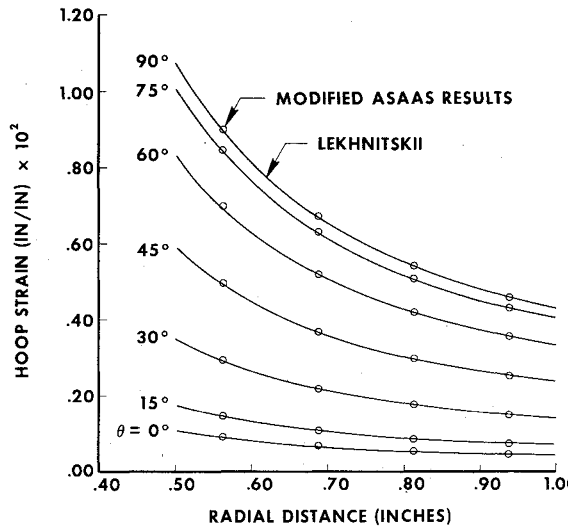

Fig. 7 Hoop strain in pressurized orthotropic disk.

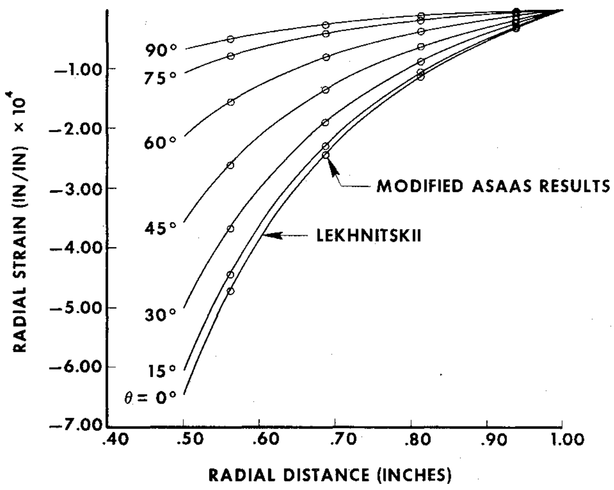

Fig. 8 Radial strain in pressurized orthotropic disk.

where a is the outer radius and b is the inner radius. It is of interest to compare the theoretical[7] stress and strain results with those predicted by ASAAS using the rectangularly orthotropic constitutive relations. For comparison purposes, an internally pressurized rectangularly orthotropic disk was investigated with the following nondimensional parameters: $a/b=2$, $E_x/p=1554$, $E_x/E_y=10$, $E_x/E_z=10$, $E_x/G_{xy}=11.1$, $\nu_{xy}=0.05$. The excellent correlation of hoop and radial stresses with only a four-element model is displayed in Figs. 5 and 6.

Inasmuch as the theoretical strain distribution is asymmetric, it is of interest to compare the ASAAS strain predictions with the theoretical[7] results. Figure 7 depicts the theoretical hoop strain plotted radially at several angular rays with respect to the material X axis. As can be seen from the figure, the predicted results of the four-element model are very close to the theoretical[7] results. The same was found for the radial strain component as indicated in Fig. 8.

B. Example 2: Thermally Loaded Disk

To verify the thermal load vector modifications, the thermal stress analysis of a solid rectangularly orthotropic disk was performed. Details of the closed-form solution were reported previously[5] along with the observation that errors on the order of 300% can be introduced in an analysis that assumes transversely isotropic properties for typical re-entry vehicle carbon-carbon materials with E_x/G_{xy} ratios of 20. The results of the 10-element modified ASAAS analyses of the thermally loaded rectangularly orthotropic disk were compared with the exact results.[5] As was the case in Example 1, all computed stress and strain components agreed well within 1% of the theoretical results for all points throughout the disk. The modified ASAAS code also has been found accurate in all other comparisons with Lekhnitskii's[7] results.

IV. Conclusions

This paper presents the modifications which can be made to the ASAAS Finite Element code to enable the analysis of three-dimensional rectangularly orthotropic materials fabricated into axisymmetric shapes and loaded axi- or asymmetrically. Since the original version of ASAAS was formulated for a polar orthotropic material, substantial modifications to the stiffness matrix and thermal load vector were necessary.

To verify the accuracy of the ASAAS code modifications, several examples were considered. Since exact answers exist for these examples, it was found that even crude finite-element models (4 to 10 elements) were able to match the known asymmetric response throughout the body within 1%. Thus, its use for problems of more complex geometry and temperature distribution appears to be justified.

Appendix

The following matrices are referred to in the text:

$$[h]=1/\lambda\begin{bmatrix} h_{11} & 0 & 0 & h_{12} & 0 & 0 & h_{13} & 0 & 0 \\ h_{21} & 0 & 0 & h_{22} & 0 & 0 & h_{23} & 0 & 0 \\ h_{31} & 0 & 0 & h_{32} & 0 & 0 & h_{33} & 0 & 0 \\ 0 & h_{11} & 0 & 0 & h_{12} & 0 & 0 & h_{13} & 0 \\ 0 & h_{21} & 0 & 0 & h_{22} & 0 & 0 & h_{23} & 0 \\ 0 & h_{31} & 0 & 0 & h_{32} & 0 & 0 & h_{33} & 0 \\ 0 & 0 & h_{11} & 0 & 0 & h_{12} & 0 & 0 & h_{13} \\ 0 & 0 & h_{21} & 0 & 0 & h_{22} & 0 & 0 & h_{23} \\ 0 & 0 & h_{31} & 0 & 0 & h_{32} & 0 & 0 & h_{33} \end{bmatrix}$$

where

$$\lambda=r_j(z_k-z_i)+r_i(z_j-z_k)+r_k(z_i-z_j)$$

$$h_{11}=r_jz_k-r_kz_j \quad h_{12}=r_kz_i-r_iz_k \quad h_{13}=r_iz_j-r_jz_i$$

$$h_{21}=z_j-z_k \quad h_{22}=z_k-z_i \quad h_{23}=z_i-z_j$$

$$h_{31}=r_k-r_j \quad h_{32}=r_i-r_k \quad h_{33}=r_j-r_i$$

and the subscripts i,j,k refer to the three nodal points of the triangular ring element proceeding counterclockwise in a right-handed r-z coordinate system.

$$g_{m_\ell}=\begin{bmatrix} 0 & \cos m_\ell\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos m_\ell\theta & 0 & 0 & 0 \\ \cos m_\ell\theta/R & \cos m_\ell\theta & Z\cos m_\ell\theta/R & 0 & 0 & 0 & m_\ell\cos m_\ell\theta/R & m_\ell\cos m_\ell\theta & m_\ell Z\cos m_\ell\theta/R \\ 0 & 0 & \cos m_\ell\theta & 0 & \cos m_\ell\theta & 0 & 0 & 0 & 0 \\ -m_\ell\sin m_\ell\theta/R & -m_\ell\sin m_\ell\theta & -m_\ell Z\sin m_\ell\theta/R & 0 & 0 & 0 & -\sin m_\ell\theta/R & 0 & -Z\sin m_\ell\theta/R \\ 0 & 0 & 0 & -m_\ell\sin m_\ell\theta/R & -m_\ell\sin m_\ell\theta & -m_\ell Z\sin m_\ell\theta/R & 0 & 0 & \sin m_\ell\theta \end{bmatrix}$$

The element stiffness matrix can be written as the double sum

$$[k]=\sum_{m_\ell=0}^{M_\ell}\sum_{m_d=0}^{M_d}[h]^T[\phi_{m_\ell m_d}][h]$$

where the 9×9 matrix $[h]$ is a function of the nodal point coordinates. Each term of the 9×9 matrix $[\phi_{m_\ell m_d}]$ involves the integration in the θ coordinate of a quadruple product of sine and/or cosine functions. The 81 terms are as follows:

$$\phi_{11}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}]\int\frac{1}{r}drdz$$

$$\phi_{21}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}+\bar{C}_{13}-m_d\bar{C}_{15}]\int drdz$$

$$\phi_{31}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}]\int\frac{z}{r}drdz$$

$$\phi_{41}=0$$

$$\phi_{51}=0$$

$$\phi_{61}=[\bar{C}_{23}-m_d\bar{C}_{25}]\int drdz$$

$$\phi_{71}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}-\bar{C}_{53}+m_d\bar{C}_{55}]\int\frac{1}{r}drdz$$

$$\phi_{81}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}]\int drdz$$

$$\phi_{91}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}-\bar{C}_{53}+m_d\bar{C}_{55}]\int\frac{z}{r}\,drdz$$

$$\phi_{12}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}+\bar{C}_{31}-m_\ell\bar{C}_{51}]\int drdz$$

$$\phi_{22}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}+\bar{C}_{11}+\bar{C}_{13}+\bar{C}_{31}-m_d\bar{C}_{15}-m_\ell\bar{C}_{51}]\int rdrdz$$

$$\phi_{32}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}+\bar{C}_{31}-m_\ell\bar{C}_{51}]\int zdrdz$$

$$\phi_{42}=0$$

$$\phi_{52}=0$$

$$\phi_{62}=[\bar{C}_{23}-m_d\bar{C}_{25}+\bar{C}_{21}]\int rdrdz$$

$$\phi_{72}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}-\bar{C}_{53}+m_d\bar{C}_{55}+m_\ell\bar{C}_{31}-\bar{C}_{51}]\int drdz$$

$$\phi_{82}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}+m_\ell\bar{C}_{31}]\int rdrdz$$

$$\phi_{92}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}-\bar{C}_{53}+m_d\bar{C}_{55}+m_\ell\bar{C}_{31}-\bar{C}_{51}]\int zdrdz$$

$$\phi_{13}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}]\int\frac{z}{r}\,drdz$$

$$\phi_{23}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}+\bar{C}_{13}-m_d\bar{C}_{15}]\int zdrdz$$

$$\phi_{33}=[\bar{C}_{33}-m_d\bar{C}_{35}-m_\ell\bar{C}_{53}+m_\ell m_d\bar{C}_{55}]\times\int\frac{z^2}{r}\,drdz+[\bar{C}_{44}]\int rdrdz$$

$$\phi_{43}=[-m_\ell\bar{C}_{64}]\int drdz$$

$$\phi_{53}=[\bar{C}_{44}-m_\ell\bar{C}_{64}]\int rdrdz$$

$$\phi_{63}=[\bar{C}_{23}-m_d\bar{C}_{25}-m_\ell\bar{C}_{64}]\int zdrdz$$

$$\phi_{73}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}-\bar{C}_{53}+m_d\bar{C}_{55}]\int\frac{z}{r}\,drdz$$

$$\phi_{83}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}]\int zdrdz$$

$$\phi_{93}=[m_\ell\bar{C}_{33}-m_\ell m_d\bar{C}_{35}-\bar{C}_{53}+m_d\bar{C}_{55}]\times\int\frac{z^2}{r}\,drdz+[\bar{C}_{64}]\int rdrdz$$

$$\phi_{14}=0$$

$$\phi_{24}=0$$

$$\phi_{34}=[-m_d\bar{C}_{46}]\int drdz$$

$$\phi_{44}=[m_\ell m_d\bar{C}_{66}]\int\frac{1}{r}\,drdz$$

$$\phi_{54}=[-m_d\bar{C}_{46}+m_\ell m_d\bar{C}_{66}]\int drdz$$

$$\phi_{64}=[m_\ell m_d\bar{C}_{66}]\int\frac{z}{r}\,drdz$$

$$\phi_{74}=0$$

$$\phi_{84}=0$$

$$\phi_{94}=[-m_d\bar{C}_{66}]\int drdz$$

$$\phi_{15}=0$$

$$\phi_{25}=0$$

$$\phi_{35}=[\bar{C}_{44}-m_d\bar{C}_{46}]\int rdrdz$$

$$\phi_{45}=[-m_\ell\bar{C}_{64}+m_\ell m_d\bar{C}_{66}]\int drdz$$

$$\phi_{55}=[\bar{C}_{44}-m_d\bar{C}_{46}-m_\ell\bar{C}_{64}+m_\ell m_d\bar{C}_{66}]\int rdrdz$$

$$\phi_{65}=[-m_\ell\bar{C}_{64}+m_\ell m_d\bar{C}_{66}]\int zdrdz$$

$$\phi_{75}=0$$

$$\phi_{85}=0$$

$$\phi_{95}=[\bar{C}_{64}-m_d\bar{C}_{66}]\int rdrdz$$

$$\phi_{16}=[\bar{C}_{32}-m_\ell\bar{C}_{52}]\int drdz$$

$$\phi_{26}=[\bar{C}_{32}-m_\ell\bar{C}_{52}+\bar{C}_{12}]\int rdrdz$$

$$\phi_{36}=[\bar{C}_{32}-m_\ell\bar{C}_{52}-m_d\bar{C}_{46}]\int zdrdz$$

$$\phi_{46}=[m_\ell m_d\bar{C}_{66}]\int\frac{z}{r}\,drdz$$

$$\phi_{56}=[m_\ell m_d\bar{C}_{66}-m_d\bar{C}_{46}]\int zdrdz$$

$$\phi_{66}=[m_\ell m_d\bar{C}_{66}]\int\frac{z^2}{r}\,drdz+[\bar{C}_{22}]\int rdrdz$$

$$\phi_{76}=[m_\ell\bar{C}_{32}-\bar{C}_{52}]\int drdz$$

$$\phi_{86}=[m_\ell C_{32}]\int rdrdz$$

$$\phi_{96}=[m_\ell\bar{C}_{32}-\bar{C}_{52}-m_d\bar{C}_{66}]\int zdrdz$$

$$\phi_{17}=[m_d\bar{C}_{33}-\bar{C}_{35}-m_\ell m_d\bar{C}_{53}+m_\ell\bar{C}_{55}]\int\frac{1}{r}\,drdz$$

$$\phi_{27}=[m_d\bar{C}_{33}-\bar{C}_{35}-m_\ell m_d\bar{C}_{53}+m_\ell\bar{C}_{55}-\bar{C}_{15}+m_d\bar{C}_{13}]\int drdz$$

$$\phi_{37}=[m_d\bar{C}_{33}-\bar{C}_{35}-m_\ell m_d\bar{C}_{53}+m_\ell\bar{C}_{55}]\int\frac{z}{r}\,drdz$$

$$\phi_{47}=0$$

$$\phi_{57}=0$$

$$\phi_{67} = [m_d\bar{C}_{23} - \bar{C}_{25}] \int drdz$$

$$\phi_{77} = [m_\ell m_d\bar{C}_{33} - m_\ell\bar{C}_{35} - m_d\bar{C}_{53} + \bar{C}_{55}] \int \frac{1}{r} drdz$$

$$\phi_{87} = [m_\ell m_d\bar{C}_{33} - m_\ell\bar{C}_{35}] \int drdz$$

$$\phi_{97} = [m_\ell m_d\bar{C}_{33} - m_\ell\bar{C}_{35} - m_d\bar{C}_{53} + \bar{C}_{55}] \int \frac{z}{r} drdz$$

$$\phi_{18} = [m_d\bar{C}_{33} - m_\ell m_d\bar{C}_{53}] \int drdz$$

$$\phi_{28} = [m_d\bar{C}_{13} - m_\ell m_d\bar{C}_{53} + m_d\bar{C}_{33}] \int rdrdz$$

$$\phi_{38} = [m_d\bar{C}_{33} - m_\ell m_d\bar{C}_{53}] \int zdrdz$$

$$\phi_{48} = 0$$

$$\phi_{58} = 0$$

$$\phi_{68} = [m_d\bar{C}_{23}] \int rdrdz$$

$$\phi_{78} = [m_\ell m_d\bar{C}_{33} - m_d\bar{C}_{53}] \int drdz$$

$$\phi_{88} = [m_\ell m_d\bar{C}_{33}] \int rdrdz$$

$$\phi_{98} = [m_\ell m_d\bar{C}_{33} - m_d\bar{C}_{53}] \int zdrdz$$

$$\phi_{19} = [m_d\bar{C}_{33} - \bar{C}_{35} - m_\ell m_d\bar{C}_{53} + m_\ell\bar{C}_{55}] \int \frac{z}{r} drdz$$

$$\phi_{29} = [m_d\bar{C}_{33} - \bar{C}_{35} - m_\ell m_d\bar{C}_{53} + m_\ell\bar{C}_{55} + m_d\bar{C}_{13} - \bar{C}_{15}] \int zdrdz$$

$$\phi_{39} = [m_d\bar{C}_{33} - \bar{C}_{35} - m_\ell m_d\bar{C}_{53} + m_\ell\bar{C}_{55}] \times \int \frac{z^2}{r} drdz + [\bar{C}_{46}] \int rdrdz$$

$$\phi_{49} = [-m_\ell\bar{C}_{66}] \int drdz$$

$$\phi_{59} = [-m_\ell\bar{C}_{66} + \bar{C}_{46}] \int rdrdz$$

$$\phi_{69} = [-m_\ell\bar{C}_{66} + m_d\bar{C}_{23} - \bar{C}_{25}] \int zdrdz$$

$$\phi_{79} = [m_\ell m_d\bar{C}_{33} - m_\ell\bar{C}_{35} - m_d\bar{C}_{53} + \bar{C}_{55}] \int \frac{z}{r} drdz$$

$$\phi_{89} = [m_\ell m_d\bar{C}_{33} - m_\ell\bar{C}_{35}] \int zdrdz$$

$$\phi_{99} = [m_\ell m_d\bar{C}_{33} - m_\ell\bar{C}_{35} - m_d\bar{C}_{53} + \bar{C}_{55}] \times \int \frac{z^2}{r} drdz + [\bar{C}_{66}] \int rdrdz$$

For a given load harmonic m_ℓ and displacement harmonic m_d, the harmonic coupling of a typical elastic coefficient C_{ij} can be expressed in compact form where a coefficient $\bar{C}_{ij}$ is defined:

for $i \le 4, j \le 4$,

$$\bar{C}_{ij} = \sum_{m_m=0}^{M_m} \sum_{n=0,2}^{4} B_n \int_0^{2\pi} \cos n\theta \cos m_m\theta \cos m_\ell\theta \cos m_d\theta d\theta$$

for $i \le 4, j \ge 5$,

$$\bar{C}_{ij} = \sum_{m_m=0}^{M_m} \sum_{n=2,4}^{4} A_n \int_0^{2\pi} \sin n\theta \cos m_m\theta \cos m_\ell\theta \sin m_d\theta d\theta$$

for $i \ge 5, j \le 4$,

$$\bar{C}_{ij} = \sum_{m_m=0}^{M_m} \sum_{n=2,4}^{4} A_n \int_0^{2\pi} \sin n\theta \cos m_m\theta \sin m_\ell\theta \cos m_d\theta d\theta$$

for $i \ge 5, j \ge 5$,

$$\bar{C}_{ij} = \sum_{m_m=0}^{M_m} \sum_{n=0,2}^{4} B_n \int_0^{2\pi} \cos n\theta \cos m_m\theta \sin m_\ell\theta \sin m_d\theta d\theta$$

Each of the four displayed integrals is simplified using trigonometric identities. For example, the integral

$$\int_0^{2\pi} \cos m_m\theta \cos n\theta \cos m_\ell\theta \cos m_d\theta d\theta$$

can be equivalently expressed as [8]

$$\tfrac{1}{8} \sum_{i=1}^{8} \int_0^{2\pi} \cos\alpha_i\theta d\theta$$

where

$$\alpha_1 = m_m + n - m_\ell - m_d$$
$$\alpha_2 = m_m - n + m_\ell + m_d$$
$$\alpha_3 = m_m + n + m_\ell - m_d$$
$$\alpha_4 = m_m - n - m_\ell + m_d$$
$$\alpha_5 = m_m + n - m_\ell + m_d$$
$$\alpha_6 = m_m - n + m_\ell - m_d$$
$$\alpha_7 = m_m + n + m_\ell + m_d$$
$$\alpha_8 = m_m - n - m_\ell - m_d$$

It is convenient to define the symbols β_i and γ_i: $\beta_i = -1$ when $\alpha_i = 0$, otherwise $\beta_i = 0$; $\gamma_i = 1$ when $\alpha_i = 0$, otherwise $\gamma_i = 0$. Thus the integral evaluation of the four cosine terms can be expressed compactly as

$$\int_0^{2\pi} \cos n\theta \cos m_m\theta \cos m_\ell\theta \cos m_d\theta d\theta = (\pi/4)(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8)$$

In a similar manner the remaining integrals can be expressed as

$$\int_0^{2\pi} \sin n\theta \cos m_m\theta \cos m_\ell\theta \sin m_d\theta d\theta = (\pi/4)(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8)$$

$$\int_0^{2\pi} \sin n\theta \cos m_m\theta \sin m_\ell\theta \cos m_d\theta d\theta$$

$$(\pi/4)(\beta_1+\gamma_2+\beta_3+\beta_4+\gamma_5+\gamma_6+\beta_7+\beta_8)$$

$$\int_0^{2\pi} \cos n\theta \cos m_m\theta \sin m_\ell\theta \sin m_d\theta d\theta$$

$$=(\pi/4)(\beta_1+\beta_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6+\beta_7+\beta_8)$$

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